ON THE NONASSOCIATIVE JEWELL-SINCLAIR THEOREM

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Abstract

A $g$-$c$-derivation is a linear mapping $D$ from a normed algebra $\mathcal{A}$ into itself such that $D(ab) = D(a)g(b) + aD(b) = D(a)b + g(a)D(b)$ for all $a, b \in \mathcal{A}$, where $g$ is a continuous linear map from $\mathcal{A}$ into itself. In this paper, we prove that any $g$-$c$-derivation on a semiprime Banach nonassociative algebra $\mathcal{A}$ is continuous if for each closed infinite dimensional ideal $I \subseteq \mathcal{A}$, there is a sequence $\{T_n\}_{n \in \mathbb{N}} \subseteq M(\mathcal{A})$ (the multiplication algebra of $\mathcal{A}$), such that the sequence $\{(T_1 T_2 \cdots T_n I)^+\}_{n \in \mathbb{N}}$ of closed right ideals of $\mathcal{A}$ is constantly decreasing. As a consequence, every $g$-$c$-derivation on nonassociative $H^*$-algebra with zero annihilator is continuous.
1. Introduction

Jewell and Sinclair in [5] obtained the continuity of derivations on certain Banach algebras known as Jewell-Sinclair theorem. In 1994, Palmer in [8] presented Jewell-Sinclair theorem in a perfect form as follows: Every derivation on a Banach algebra \( A \) is continuous if \( A \) satisfies:

(i) \( A \) has no nonzero finite dimensional nilpotent ideals.

(ii) For each closed infinite dimensional ideal \( I \) of \( A \), there is a sequence \( \{a_n\}_{n \in \mathbb{N}} \subseteq A \) such that the sequence \( \{(a_1a_2\cdots a_nI)^-\}_{n \in \mathbb{N}} \) of closed right ideals of \( A \) is constantly decreasing.

In [6], they generalized the above Palmer presentation of Jewell-Sinclair theorem in the nonassociative setting as follows: Let \( A \) be a semiprime Banach algebra not necessarily associative, such that for each closed infinite dimensional ideal \( I \subseteq A \), there is a sequence \( \{T_n\}_{n \in \mathbb{N}} \subseteq M(A) \) (the multiplication algebra of \( A \)), such that the sequence \( \{(T_1T_2\cdots T_nI)^-\}_{n \in \mathbb{N}} \) of closed right ideals of \( A \) is constantly decreasing, then any derivation on \( A \) is continuous. Also, they used this result and Villena’s lines proof in [11, Theorem 4] to prove that every derivation on nonassociative \( H^* \)-algebra with zero annihilator is continuous.

In this paper, we will prove that every \( g \)-c-derivation on nonassociative \( H^* \)-algebra with zero annihilator is continuous via nonassociative Jewell-Sinclair theorem. So our purpose is the following theorems:

**Theorem A.** Let \( A \) be a semiprime complete normed algebra such that for each closed infinite dimensional ideal \( I \subseteq A \), there is a sequence \( \{T_n\}_{n \in \mathbb{N}} \subseteq M(A) \) such that the sequence \( \{(T_1T_2\cdots T_nI)^-\}_{n \in \mathbb{N}} \) of closed right ideals of \( A \) is constantly decreasing. Then any \( g \)-c-derivation on \( A \) is continuous.
Theorem B. Every g-c-derivation on nonassociative $H^*$-algebra $A$ with zero annihilator is continuous.

Following [7], we define a g-c-derivation as follows: Let $A$ be a normed algebra, a linear mapping $D$ from $A$ into itself is called a g-c-derivation if $D(ab) = D(a)g(b) + aD(b) = D(a)b + g(a)D(b)$ for all $a, b \in A$, where $g$ is continuous linear map from $A$ into itself, note that if $g$ is the identity map, then $D$ is the usual derivation. Recall from [10] that, if $X$ and $Y$ are normed spaces and if $T : X \to Y$ is linear map, then separating subspace $S(T)$ of $T$ is define as follows: $S(T) = \{ y \in Y : \exists$ a sequence $\{x_n\}$ in $X, n \in \mathbb{N}$ with $\lim x_n = 0$ and $\lim T(x_n) = y \}$. The separating space $S(T)$ is a closed linear subspace of $Y$. Also, recall from [1] that an annihilator of an algebra $A$ (denoted by Ann$(A)$) is defined as the set of those $a$ in $A$ satisfying $ab = ba = 0$ for every $b \in A$. An algebra $A$ is semiprime if for any ideal $I$ of $A$ such that $I^2 = 0$, then $I = 0$ and $A$ is prime, if for any two ideals $I$ and $J$ of $A$ such that $I \cap J = 0$, then either $I = 0$ or $J = 0$. Also, if $A$ has nonzero product and has no nonzero proper closed ideals, then $A$ is topologically simple. The multiplication algebra of $A$ denoted by $M(A)$ is defined as a subalgebra of $L(A)$ (the algebra of all linear mapping on $A$) generated by $L_a, R_a,$ and $Id_A$, which is left, right, and identity multiplication operators, respectively. An involution of an algebra $A$ is a mapping $x \to x^*$ of $A$ into $A$ such that for all $x, y \in A, \alpha \in \mathbb{C}$ (complex field) the mapping $^*$ satisfies the following conditions:

(i) $(x + y)^* = x^* + y^*$; (ii) $(ax)^* = a^*x^*$; (iii) $(x^*)^* = x$; (iv) $(xy)^* = y^*x^*$.

A nonassociative $H^*$-algebra is an algebra $A$ with algebra involution $^*$, whose underlying vector space is a Hilbert space satisfying $\langle ab, c \rangle = \langle a, cb^* \rangle = \langle b, a^*c \rangle$. 
Finally, we recall that a prime algebra $A$ over a field $\mathbb{C}$ is said to be centrally closed, if for every nonzero ideal $I$ of $A$ and for every linear mapping $f : I \to A$ with $f(ax) = af(x)$ and $f(xa) = f(x)a$, for all $a \in A$ and $x \in I$, then there exists $\lambda \in \mathbb{C}$ such that $f(x) = \lambda x$ for all $x \in I$ (see [4]). From now on, in this paper, all algebra are not necessarily associative over a complex field.

2. Proof of Theorem A

For a Banach space $X$, we denote by $BL(X)$ (the Banach space of all bounded linear mapping on $X$). We begin this section by the following results:

**Lemma 2.1** ([8, Lemma 6.1.17]). Let $X$ and $Y$ be Banach spaces. Let $\{S_n\}_{n \in \mathbb{N}} \subseteq BL(X)$ and $\{R_n\}_{n \in \mathbb{N}} \subseteq BL(Y)$ and $T \in L(X, Y)$ satisfy $TS_n - R_n T \in BL(X, Y)$ for all $n \in \mathbb{N}$. Then, there is an integer $k$ such that $(R_1 R_2 \cdots R_n S(T))^- = (R_1 R_2 \cdots R_k S(T))^- \quad \text{for all } n \geq k.

**Lemma 2.2.** If $A$ is a normed algebra and if $D$ is $g$-c-derivation on $A$. Then $S(D)$ is closed ideal of $A$.

**Proof.** It is clear that $S(D)$ is a closed subspace of $A$. Let $b \in S(D)$, there exists a sequence $\{a_n\}$ in $A$ such that lim $a_n = 0$ and lim $D(a_n) = b$. For all $a \in A$, we have lim $a_n a = 0$. Since $g$ is continuous, it follows that lim $g(a_n) = 0$. Now lim $D(a_n a) = lim D(a_n) a +$ lim $g(a_n) D(a) = ba$. Therefore $ba \in S(D)$. Similarly, $ab \in S(D)$. This complete the proof. 

**Lemma 2.3.** If $A$ is a normed algebra and if $D$ is $g$-c-derivation on $A$, then $DT - TD \in BL(A)$, for all $T \in M(A)$. 

Proof. Let \( \mathcal{F} = \{ T \in BL(A) : DT - TD \in BL(A) \} \). It is clear that \( \mathcal{F} \) is subspace of \( BL(A) \). For \( T_1, T_2 \in \mathcal{F} \), we have \( DT_1T_2 - T_1T_2D = (DT_1 - T_1D)T_2 + T_1(DT_2 - T_2D) \). Therefore \( DT_1T_2 - T_1T_2D \in BL(A) \), that is, \( T_1T_2 \in \mathcal{F} \). As a consequence, \( \mathcal{F} \) is a subalgebra of \( BL(A) \). Since \( L_a, R_a \), and \( Id_A \) are in \( \mathcal{F} \), it follows that \( \mathcal{F} = M(A) \). \( \square \)

The proof of the following theorem is similar to that given in [6, Theorem 2-1].

**Theorem A.** Let \( A \) be a semiprime complete normed algebra such that for each closed infinite dimensional ideal \( I \subseteq A \), there is a sequence \( \{T_n\}_{n \in \mathbb{N}} \subseteq M(A) \) such that the sequence \( \{(T_1T_2 \cdots T_nI)\}_{n \in \mathbb{N}} \) of closed right ideals of \( A \) is constantly decreasing. Then any \( g \)-c-derivation on \( A \) is continuous.

**Proof.** Let \( D \) be a \( g \)-c-derivation on \( A \). By Lemma 2.2, \( S(D) \) is closed ideal of \( A \). If \( S(D) \) is infinite dimensional, then by assumption, there is a sequence \( \{T_n\}_{n \in \mathbb{N}} \subseteq M(A) \) such that the sequence \( \{(T_1T_2 \cdots T_nS(D))\}_{n \in \mathbb{N}} \) is constantly decreasing. Applying Lemmas 2.1 and 2.3 by setting: \( X = Y = A, T = D, R_n = T_n = S_n \), we get, there exist a natural number \( k \in \mathbb{N} \) such that \((T_1T_2 \cdots T_nS(D)) = (T_1T_2 \cdots T_kS(D)) \) for all \( n \geq k \). This condition implies that \( \{T_n\}_{n \in \mathbb{N}} \) not constantly decreasing, a contradiction. So, \( S(D) \) must be finite dimensional. Note that, if \( S(D) \) is finite dimensional, then \( D|_{S(D)} \) is continuous. Now we claim that \( S(D) = \{0\} \). Let \( a, b \in S(D) \), then there exists a sequence \( \{a_n\} \) in \( A \) such that \( \lim a_n = 0 \) and \( \lim D(a_n) = a \). Now since \( \lim a_n = 0 \), then \( \lim a_nb = 0 \) and since \( g \) is continuous, then \( \lim g(a_n) = 0 \). Since \( S(D) \) is an ideal of
A and \( b \in S(D) \), then \( a_n b \in S(D) \) but \( D|_{S(D)} \) is continuous, then \( \lim D(a_n b) = 0 \). Since \( D(a_n b) = D(a_n) b + g(a_n) D(b) \), then \( \lim D(a_n b) = ab \), this implies that \( ab = 0 \). Therefore \( S(D)^2 = \{0\} \) and since \( A \) is semiprime, we have \( S(D) = 0 \) and by closed-graph theorem, we obtain that \( D \) is continuous. As required. \( \square \)

3. Proof of Theorem B

In this section, we need the following lemmas before we give our proof of Theorem B:

**Lemma 3.1 ([11]).** Let \( A \) be a centrally closed prime algebra such that \( \dim(T(A)) > 1 \) for all nonzero \( T \) in the multiplication algebra \( M(A) \) of \( A \). Then, there is a sequence \( \{b_n\}_{n \in \mathbb{N}} \) in \( A \) and \( \{T_n\}_{n \in \mathbb{N}} \) in \( M(A) \) such that \( T_n T_{n-1} \cdots T_1 b_n \neq 0 \) and \( T_{n+1} \cdots T_1 b_n = 0 \) for all \( n \in \mathbb{N} \).

**Lemma 3.2 ([3]).** Every \( H^* \)-algebra with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals, and these are topologically simple \( H^* \)-algebra.

**Lemma 3.3 ([2]).** Every topologically simple \( H^* \)-algebra is centrally closed prime algebra.

**Lemma 3.4 ([9]).** Let \( A \) be an algebra, and assume the existence of a non-degenerate symmetric associative bilinear form \( \langle \cdot, \cdot \rangle \) on \( A \). Then we have

(i) there exist a unique linear algebra involution \( \# \) on the multiplication algebra \( M(A) \) of \( A \) satisfying \( L^\#_d = R_d \) and \( R^\#_d = L_d \) for all \( d \in A \);

(ii) for all \( x, y \in A \) and \( T \in M(A) \). The equality \( \langle Tx, y \rangle = \langle x, T^\# y \rangle \) holds.
Lemma 3.5. Let $D$ be a g-c-derivation on topologically simple complete normed algebra $A$ and suppose that there exists a nonzero $T \in M(A)$ with finite dimensional range satisfying $DT = TD$. Then $D$ is continuous.

Proof. Let $x \in S(D)$, then there exists a sequence $\{a_{n}\}$ in $A$ such that $\lim na_{n} = 0$ and $\lim Da_{n} = x$. Now, since $TD$ is continuous and $\lim a_{n} = 0$, then $\lim TD(a_{n}) = 0$, hence $T(x) = 0$. Therefore, $T(S(D)) = \{0\}$. But $S(D)$ is closed ideal of $A$, then $T(S(D)) = \{0\}$. Since $A$ topologically simple, we have $T(A) = \{0\}$. This is a contradiction because $T$ is nonzero. Then $S(D) = \{0\}$. Thus $D$ is continuous by the closed-graph theorem. \hfill \square

A well-known result is due to Villena [11, Theorem 4], which states that: If $A$ is $H^*$-algebra with zero annihilator, then any derivation on $A$ is continuous. We present this result in a more general setting and we will use the nonassociative Jewell-Sinclair theorem in our proof, as we will see that in the following theorem:

Theorem B. Every g-c-derivation on $H^*$-algebra $A$ with zero annihilator is continuous.

Proof. Let $D$ be a g-c-derivation on $H^*$-algebra $A$ with zero annihilator. At first, we assume that $A$ is topologically simple. Applying Lemma 3.3, we have $A$ is a centrally closed prime algebra. Now, $M(A)$ satisfying one of the following cases:

(i) There exist an element $T$ in $M(A)$ such that $T$ has finite dimensional range and $DT = TD$.

(ii) Every element in $M(A)$ has infinite dimensional range or $DT \neq TD$ for all $T$ in $M(A)$. 
First case. If (i) is true, then by using Lemma 3.5, \( D \) is continuous.

Second case. If (ii) is true, since every element in \( M(A) \) has infinite dimensional range, it follows that from Lemma 3.1, there exists a sequence \( \{ C_n \}_{n \in \mathbb{N}} \subseteq A \) and \( \{ T_n \}_{n \in \mathbb{N}} \subseteq M(A) \) such that for all \( n \in \mathbb{N} \),

\[
T_{n+1}T_n \cdots T_1C_n = 0, \\
T_nT_{n-1} \cdots T_1C_n \neq 0.
\]

Taking into account that every topologically simple \( H^* \)-algebra contains a non-degenerate symmetric associative bilinear continuous form \( \langle \ldots \rangle \) with an algebra involution \( # \) on \( M(A) \) satisfying:

\[
L_b^# = R_b, \quad R_b^# = L_b \quad \text{for all } b \in A.
\]

Now, if we suppose that \( n \in \mathbb{N} \) is a positive integer number satisfying:

\[
\overline{T_1^# \cdots T_n^#(A)} = \overline{T_1^# \cdots T_{n+1}^#(A)},
\]

then we have a contradiction. Indeed,

\[
0 = \langle A, T_{n+1} \cdots T_1(C_n) \rangle = \langle A, ((T_{n+1} \cdots T_1)^#)^#(C_n) \rangle = \langle (T_{n+1} \cdots T_1)^#(A), C_n \rangle \\
= \langle T_1^# \cdots T_{n+1}^#(A), C_n \rangle = \overline{T_1^# \cdots T_{n+1}^#(A), C_n} = \langle T_1^# \cdots T_n^#(A), C_n \rangle \\
= \langle T_1^# \cdots T_n^#(A), C_n \rangle = \langle A, ((T_n \cdots T_1)^#)^#(C_n) \rangle = \langle A, T_n^# \cdots T_1(C_n) \rangle.
\]

Therefore, for every \( n \in \mathbb{N} \), there exists a sequence \( \left\{ \overline{T_1^# \cdots T_n^#(A)} \right\}_{n \in \mathbb{N}} \) of closed right ideals of \( A \) constantly decreasing. Since \( A \) is semiprime, applying (Theorem A) we get \( D \) is continuous. In order to obtain the general case of the proof, assume that \( A \) has zero annihilator. Now, let \( M \) be a minimal closed ideal of \( A \). If the inclusion \( D(M) \subseteq M \) is not true, then there exists a nonzero minimal closed \( J \) of \( A \) such that \( D(M) \subseteq J \), and \( J \cap M = \{0\} \). This is a contradiction because \( A \) is semiprime. Then the inclusion must be \( D(M) \subseteq M \), or \( D(M) \subseteq J \cap M \subseteq M \). Since \( M \) is
topologically simple by Lemma 3.2, it follows from the first part of the proof that $D$ is continuous on $M$. Let $a \in S(D)$, then there exists a sequence $\{a_n\}$ in $A$ such that $\lim a_n = 0$ and $\lim Da_n = a$. Now, for all $y \in M$, we have $\lim ya_n = 0$, since $M$ is an ideal, then $ya_n \in M$. Since $D$ is continuous on $M$, it follows that $\lim D(ya_n) = 0$. Now, $D(ya_n) = D(y)g(a_n) + yD(a_n)$, then $\lim D(ya_n) = ya$. Therefore $ya = 0$. Similarly, $ay = 0$. For all $a \in S(D)$ and $y \in M$, then $0 = MS(D) = S(D)M$, for each minimal closed ideal $M$ of $A$. Therefore $0 = AS(D) = S(D)A$, which implies that $S(D) \subseteq \text{Ann}(A) = \{0\}$. Thus $D$ is continuous by the closed-graph theorem.

\[\square\]

References


